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On a discretised spectral approximation in neutron transport theory

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Abstract. An approximation method significantly different from existing ones is developed in this paper for the one-speed neutron transport equation. The theory is based on the singular eigenfunction method, and is founded on the new concepts of (i) a rational function approximation of the Case singular eigenfunction and (ii) discretisation of the continuous spectrum (0, 1) of the half-range transport problem using the roots of a set of polynomials orthogonal in (0, 1) with respect to a simple closed-form accurate approximation of the half-range weight function $W(\mu)$. A sample numerical example is given to illustrate the application of the theory.

1. Introduction

The neutron transport equation in one-dimensional plane geometry is an interesting representative of a class of linear equations in mathematical physics where the associated eigenspectrum is the union of discrete and continuous parts. The general solution of such an equation is therefore written as the sum of an integral over the continuous spectrum and a sum over the discrete. Among the exact methods to obtain this representation for the transport equation, that of Case [1] (the Weiner-Hopf approach is an equivalent alternate decomposition) is an expansion method in terms of a complete set of spectral functions of the inverse transport operator, where one member of the complete family is a singular distribution or generalised function. The solution procedure is based on Muskhelishvili's treatment of singular equations of Cauchy type, which is derived from the theory of analytic functions of a complex variable.

Though the singular eigenfunction method is an elegant formalism, it can be considered to be a formal method as the exact solution it produces is, in general, unsuited for numerical calculations. This is due to the appearance of a 'transient integral' term involving the singular eigenfunction and expansion coefficients that satisfy singular integral equations. This means that the resulting 'transient integral' cannot be expressed in a closed form, a feature that we shall refer to as the 'transient integral problem' of the neutron transport equation. Of the two noteworthy recent developments that circumvent the transient integral problem without actually solving it, the F_N method [2] projects Case's half-range solution onto a subspace of the full-range problem and uses numerical methods to solve a resulting Fredholm integral equation. This conceptually simple method has been utilised by Siewert and coworkers [2] for a variety of test problems to give very accurate numerical results. The less

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investigated TP_N method [3], on the other hand, replaces the transient integral by a sum and obtains the superposition coefficients of the solution by generalising the Marshak boundary conditions so as to obtain the exact extrapolated endpoint in the standard Milne problem. The series replacement of the transient integral is, as usual, in terms of classical orthogonal polynomials and it needs to be pointed out that such an expansion of singular distributions is only a formal represntation of the generalised function (GF)—in the case of the delta distribution, for example, this is merely a statement of the completeness of the polynomials—rather than an expansion theorem for the GF itself. This basic limitation has led to the introduction [4, 5] of the significantly new concept of a rational function approximation of the singular eigenfunction $\phi(\nu, \mu)$. As explained in [5], rational function approximations for the Dirac delta and Cauchy principal value exist, and it is the purpose of this paper to demonstrate how the transient integral problem can be resolved in terms of these rational Cauchy sequences. As the concern here is with half-range problems only, we will discretise the continuous spectrum (0, 1) at the zeros of a new set of orthogonal polynomials [6] with respect to a simple and accurate closed-form rational approximation of the Case half-range weight function $W(\mu)$, $0 \leq \mu \leq 1$ [4, 6]. In summary, the approach is to discretise the continuous spectrum and use this with a regular, rational function approximation of the singular eigenfunction $\phi(\nu, \mu)$ to obtain the desired solution of the transport equation.

2. The theory

This section is divided into several subsections that introduce the various components of the theory of the discretised spectral approximation.

2.1. Rational function approximation of the singular eigenfunction

The rational function approximation of Case's singular eigenfunction

$$\phi(\nu, \mu) = \frac{c\nu}{2} P\frac{1}{\nu - \mu} + \lambda(\nu)\delta(\nu - \mu)$$

is given by [5]

$$\phi_\varepsilon(\nu, \mu) = \frac{c\nu}{2}\frac{\nu - \mu}{(\nu - \mu)^2 + \varepsilon^2} + \frac{\lambda_\varepsilon}{\pi_\varepsilon}\frac{\varepsilon}{(\nu - \mu)^2 + \varepsilon^2} \tag{2.1}$$

where the normalisation $\int_{-1}^{1} \phi_\pi(\nu, \mu)\, d\mu = 1$ gives

$$\frac{\lambda_\varepsilon}{\pi_\varepsilon} = \left[1 - \frac{c\nu}{4}\ln\left(\frac{(1+\nu)^2 + \varepsilon^2}{(1-\nu)^2 + \varepsilon^2}\right)\right]\left[\tan^{-1}\left(\frac{1+\nu}{\varepsilon}\right) + \tan^{-1}\left(\frac{1-\nu}{\varepsilon}\right)\right]^{-1} \tag{2.2}$$

$$\to \frac{1}{\pi}\left[1 - \frac{c\nu}{2}\ln\left(\frac{1+\nu}{1-\nu}\right)\right] = \frac{1}{\pi}\lambda(\nu) \qquad \varepsilon \to 0. \tag{2.3}$$

From the details of [5] we recall the following salient features:

(i) for the present purpose a GF is an element of the completion of the space of integrable functions $X(I)$ with convergence defined with respect to the class of test functions $C_0^\infty(I)$, i.e. a GF on I is an equivalence class, or generalised limit, of Cauchy sequences in $X(I)$ [7], and

(ii) the convergence of $\phi_\varepsilon(\nu, \mu)$ to $\phi(\nu, \mu)$ for arbitrary ν is in the above distributional sense in L_p, $1 < p < \infty$, i.e.

$$\|\phi(\nu, \mu) - \phi_\varepsilon(\nu, \mu)\|_p \to 0 \qquad \varepsilon \to 0;\ 1 < p < \infty. \tag{2.4}$$

Remark. It is sufficient to consider the test class to be $C_0^\infty(I)$ rather than the more restrictive $\mathcal{D}(I)$ necessary in the Schwartz definition of a distribution as a continuous linear functional on $\mathcal{D}(I)$ [7].

With the rational function approximation chosen as in (2.1), it is shown in the following that the solution of the transport equation can be expressed as

$$\psi_{\varepsilon N}(x, \mu) = a(\nu_0)\, \mathrm{e}^{-x/\nu_0}\phi(\nu_0, \mu) + \sum_{i=1}^{N} a(\nu_i)\, \mathrm{e}^{-x/\nu_i}\phi_\varepsilon(\nu_i, \mu) \tag{2.5}$$

where the choice of the quadrature nodes $\{\nu_i\}_1^N$ is explained below.

2.2. Quadrature scheme of the transient integral

In writing (2.5), we have replaced the transient integral $\int_0^1 A(\nu)\, \mathrm{e}^{-x/\nu}\phi(\nu, \mu)\, \mathrm{d}\nu$ of the exact solution by the sum

$$\sum_{i=1}^{N} a(\nu_i)\, \mathrm{e}^{-x/\nu_i}\phi_\varepsilon(\nu_i, \mu). \tag{2.6}$$

As is well known, the evaluation of the coefficients $A(\nu)$ leads to a singular integral equation of the Cauchy type for $A(\nu)$. However, in practice, this solution of an integral equation is unnecessary as both $a(\nu_0)$ and $A(\nu)$ can be simply obtained by the use of the half-range orthogonality relations [8, 1]. In order that (2.6) replace the exact transient integral, it is necessary and sufficient, therefore, that the solution of the Fredholm integral equation (FIE) with (2.1) as the kernel tends as $\varepsilon \to 0$ to the solution of the corresponding singular integral equation (SIE) with $\phi(\nu, \mu)$ as kernel. To show this we note the following results.

Result 1. Let

$$H\phi(t) = \frac{1}{\pi} \int\!\!\!\!\!\!- _{-1}^{1} \frac{\phi(s)}{s - t}\, \mathrm{d}s \qquad H_\varepsilon\phi(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{s - t}{(s - t)^2 + \varepsilon^2}\, \phi(s)\, \mathrm{d}s$$

and

$$-\Delta\phi(t) = \int_{-1}^{1} \delta(t - s)\phi(s)\, \mathrm{d}s \qquad -\Delta_\varepsilon\phi(t) = \frac{\varepsilon}{\pi} \int_{-1}^{1} \frac{1}{(s - t)^2 + \varepsilon^2}\, \phi(s)\, \mathrm{d}s.$$

Then it is known [5] that $H\phi \in L_p(-1, 1)$, $\Delta\phi \in L_p(-1, 1)$ for $p > 1$. Also $\|H\phi - H_\varepsilon\phi\|_p \to 0$ and $\|\Delta\phi - \Delta_\varepsilon\phi\|_p \to 0$ $\varepsilon \to 0$ for all $\phi \in L_p(-1, 1)$, $p > 1$. This implies, by the Banach-Steinhaus theorem, that H_ε and Δ_ε are uniformly bounded, i.e. $\|H_\varepsilon\|_p < M_1$, $\|\Delta_\varepsilon\|_p < M_2$. Let us now consider the SIE and their corresponding Fredholm approximations:

$$u(t) = f(t) + \Delta u(t) \qquad v(t) = f(t) + Hv(t) \qquad \text{SIE}$$

$$u_\varepsilon(t) = f(t) + \Delta_\varepsilon u_\varepsilon(t) \qquad v_\varepsilon(t) = f(t) + H_\varepsilon v_\varepsilon(t). \qquad \text{FIE}$$

The operators H_ε and Δ_ε are compact in $L_p(-1,1)$, $C(-1,1)$, and hence the FIE can be solved by the degenerate kernel method. Denoting the SIE and FIE in the forms $(I-K)y=f$ and $(I-K_\varepsilon)y_\varepsilon=f$ respectively, using the identity

$$T_\varepsilon^{-1}-S^{-1}=T_\varepsilon^{-1}(S-T_\varepsilon)S^{-1}$$

and assuming the solution of the SIE to exist (i.e. assuming S^{-1} to exist) we obtain the second result.

Result 2. The SIE and FIE have solutions $y=S^{-1}f$ and $y_\varepsilon=T_\varepsilon^{-1}f$ respectively, and

$$\|y_\varepsilon-y\|\leqslant\|T_\varepsilon^{-1}\|\,\|(K-K_\varepsilon)y\|.$$

Proof. With $S=I-K$, $T_\varepsilon=I-K_\varepsilon$ and the identity above, we get

$$\begin{aligned}\|y_\varepsilon-y\|&=\|(I-K_\varepsilon)^{-1}f-(I-K)^{-1}f\|\\&\leqslant\|(I-K_\varepsilon)^{-1}(K_\varepsilon-K)(I-K)^{-1}f\|\\&\leqslant\|(I-K_\varepsilon)^{-1}\|\,\|(K_\varepsilon-K)y\|.\end{aligned}$$

Hence by result 1, $y_\varepsilon\to y$ strongly as $\varepsilon\to 0$.

To determine the best possible (i.e. Gaussian) quadrature nodes in (2.6), we recognise the half-range weight $W(\mu)^1$ to be the natural weight function of the half-range problem. Writing

$$\psi_\varepsilon(x,\mu)=a(\nu_0)\,\mathrm{e}^{-x/\nu_0}\phi(\nu_0,\mu)+\int_0^1 A_\varepsilon(\nu)\,\mathrm{e}^{-x/\nu}\phi_\varepsilon(\nu,\mu)\,\mathrm{d}\nu \tag{2.7}$$

as

$$\psi_\varepsilon(x,\mu)=a(\nu_0)\,\mathrm{e}^{-x/\nu_0}\phi(\nu_0,\mu)+\int_0^1 W(\nu)\bar{a}_\varepsilon(\nu)\,\mathrm{e}^{-x/\nu}\phi_\varepsilon(\nu,\mu)\,\mathrm{d}\nu$$

where $A_\varepsilon(\nu)=W(\nu)\bar{a}_\varepsilon(\nu)$, we see that

$$W(\nu)\bar{a}_\varepsilon(\nu)\,\mathrm{e}^{-x/\nu}\phi_\varepsilon(\nu,\mu)=W(\nu)\sum_{i=1}^{N}\bar{a}_\varepsilon(\nu_i)\,\mathrm{e}^{-x/\nu_i}\phi_\varepsilon(\nu_i,\mu)l_i(\nu)$$

is the Lagrange interpolation formula for the expression on the LHS, where

$$l_i(\nu)=\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{\nu-\nu_j}{\nu_i-\nu_j}$$

are the fundamental polynomials of Lagrange interpolation, and $\{\nu_i\}_1^N$ are the zeros of the set of orthogonal polynomials WRT $W(\nu)$, $0\leqslant\nu\leqslant 1$, i.e. these are the Gaussian nodes of the half-range problem. With this interpolation, (2.7) becomes

$$\psi_{\varepsilon N}(x,\mu)=a(\nu_0)\,\mathrm{e}^{-x/\nu_0}\phi(\nu_0,\mu)+\sum_{i=1}^{N}a(\nu_i)\,\mathrm{e}^{-x/\nu_i}\phi_\varepsilon(\nu_i,\mu) \tag{2.8}$$

where

$$a(\nu_i)=\bar{a}_\varepsilon(\nu_i)\int_0^1 W(\nu)l_i(\nu)\,\mathrm{d}\nu \qquad \bar{a}_\varepsilon(\nu_i)=A_\varepsilon(\nu_i)/W(\nu_i)$$

which is (2.5).

2.3. Simple, accurate, closed-form approximation of $W(\mu)$ [4, 6]

Because of its essentially transcendental nature, arbitrary integrations involving $W(\mu)$, as in the expression for $a(\nu_i)$ above, cannot be carried out straightforwardly. To overcome this, the following simple and accurate closed-form rational approximation of the lowest order was introduced [4] for $W(\mu)$:

$$W^{(0)}(\mu) = \frac{c}{2\Omega^{(0)}(1-c)} \frac{\mu(\alpha+\mu)}{\nu_0+\mu} \tag{2.9}$$

where

$$\Omega^{(0)} = 0.5(\Omega_+ + \Omega_-) \qquad \alpha = \nu_0(1-c)^{1/2}. \tag{2.10}$$

with Ω_+ and Ω_- as given in [4]. Equation (2.9) is a one-parameter approximation of the half-range weight function where the parameter $\Omega^{(0)}$ is given by (2.10). For higher-order approximations we use

$$\Omega^{(i+1)}(-\mu) = 1 - \frac{c\nu_0^2}{2}\mu \int_0^1 \frac{1 - t^2/\nu_0^2(1-c)}{(\nu_0^2 - t^2)(\mu+t)\Omega^{(i)}(-t)}\,\mathrm{d}t \tag{2.11}$$

$$X^{(i)}(-\mu) = (\alpha+\mu)^{-1}\Omega^{(i)}(-\mu) \tag{2.12}$$

to give the explicit result to order one

$$\Omega^{(1)}(-\mu) = 1 - \frac{c\nu_0^2}{2\Omega^{(0)}}\mu\mathscr{S}^{(0)}(\mu) \tag{2.13}$$

$$\mathscr{S}^{(0)}(\mu) = \alpha_1(\mu)\ln\left(\frac{1+\nu_0}{\nu_0}\right) + \alpha_2(\mu)\ln\left(\frac{\nu_0-1}{\nu_0}\right) + \alpha_3(\mu)\ln\left(\frac{1+\mu}{\mu}\right)$$

where

$$\alpha_1(\mu) = \frac{c}{2(1-c)}\frac{1}{\nu_0(\nu_0-\mu)}$$

$$\alpha_2(\mu) = \frac{c}{2(1-c)}\frac{1}{\nu_0(\nu_0+\mu)}$$

$$\alpha_3(\mu) = \frac{1}{\nu_0^2(1-c)} - \frac{c}{1-c}\frac{1}{\nu_0^2-\mu^2}.$$

Hence

$$X^{(0)}(-\mu) = (\alpha+\mu)^{-1}\Omega^{(0)} \tag{2.14a}$$

and

$$X^{(1)}(-\mu) = (\alpha+\mu)^{-1}\left(1 - \frac{c\nu_0^2}{2\Omega^{(0)}}\mu\mathscr{S}^{(0)}(\mu)\right) \tag{2.14b}$$

are the two lowest-order approximations of the X function. The reliability of these expressions depend on $\Omega^{(0)}$, i.e. on equation (2.10), obtained from the orthogonality integral involving $\phi(\pm\nu_0, \mu)$ [4]. Denoting

$$E_X^{(i)}(-\mu) = \frac{X(-\mu) - X^{(i)}(-\mu)}{X(-\mu)} \qquad i = 0, 1$$

to be the errors in $X^{(0)}$, $X^{(1)}$, we show in table 1 the variation in $E^{(0)}$, $E^{(1)}$ for some values of μ. Introducing (2.14*b*) in the expression for $W(\mu)$ [1], gives a first-order approximation, $W^{(1)}(\mu)$. The form of $W^{(0)}(\mu)$ is particularly useful, however, as unlike

Table 1. $\Omega^{(0)}$, $\Omega^{(1)}$, $X^{(0)}$, $X^{(1)}$ for different μ.

μ	$\Omega^{(1)}(-\mu)$	$X^{(0)}(-\mu)$	$X^{(1)}(-\mu)$	$E_x^{(0)}\times 10^3$	$E_x^{(1)}\times 10^3$
$c = 0.1$, $\Omega_+ = 0.989\,872$, $\Omega_- = 0.991\,211$, $\Omega^{(0)} = 0.990\,542$					
0.2	0.990 533	0.862 328	0.862 320	0.005 725	0.014 837
0.4	0.989 414	0.734 451	0.733 615	−1.127 328	0.012 388
0.6	0.989 538	0.639 602	0.638 955	−1.002 544	0.011 478
0.8	0.990 010	0.566 450	0.566 146	−0.528 320	0.009 256
1.0	0.990 574	0.508 313	0.508 330	0.036 578	0.004 260
$c = 0.3$, $\Omega_+ = 0.971\,659$, $\Omega_- = 0.975\,391$, $\Omega^{(0)} = 0.973\,525$					
0.2	0.972 857	0.937 136	0.936 493	−0.566 059	0.120 419
0.4	0.970 094	0.785 842	0.783 073	−3.434 005	0.102 336
0.6	0.970 711	0.676 609	0.674 653	−2.811 247	0.087 443
0.8	0.972 205	0.594 037	0.593 231	−1.281 995	0.075 436
1.0	0.973 883	0.529 426	0.529 622	0.435 605	0.066 792
$c = 0.5$, $\Omega_+ = 0.959\,664$, $\Omega_- = 0.959\,433$, $\Omega^{(0)} = 0.959\,548$					
0.2	0.957 607	1.022 439	1.020 370	−1.731 008	0.295 706
0.4	0.954 159	0.842 826	0.838 092	−5.396 142	0.250 260
0.6	0.955 615	0.716 889	0.713 951	−3.902 498	0.212 156
0.8	0.958 212	0.623 695	0.622 826	−1.211 899	0.182 778
1.0	0.960 965	0.551 944	0.552 759	1.635 921	0.161 727
$c = 0.7$, $\Omega_+ = 0.948\,433$, $\Omega_- = 0.947\,192$, $\Omega^{(0)} = 0.947\,812$					
0.2	0.944 435	1.100 835	1.096 913	−3.052 502	0.521 286
0.4	0.940 972	0.893 325	0.886 878	−6.823 075	0.443 135
0.6	0.943 448	0.751 639	0.748 178	−4.246 147	0.377 526
0.8	0.947 139	0.648 745	0.648 284	−0.383 757	0.326 466
1.0	0.950 883	0.570 630	0.572 478	3.515 971	0.287 488
$c = 0.9$, $\Omega_+ = 0.938\,079$, $\Omega_- = 0.937\,206$, $\Omega^{(0)} = 0.937\,643$					
0.2	0.932 814	1.169 355	1.163 332	−4.383 608	0.789 201
0.4	0.929 728	0.935 915	0.928 015	−7.833 690	0.672 875
0.6	0.933 286	0.780 169	0.776 544	−4.090 807	0.574 355
0.8	0.938 021	0.668 863	0.669 132	0.901 236	0.498 273
1.0	0.942 665	0.585 351	0.588 487	5.766 180	0.440 208

the higher-order more involved approximations, it can be used as a weight function in the evaluation of integrals without undue algebraic complexity.

To verify how accurately this may actually be achieved, we evaluated the following test integrals:

$$\mathrm{I}_{\pm}^{(0)}(\nu) = \int_0^1 W^{(0)}(\mu)\phi(\pm\nu_0, \mu)\phi(\nu, \mu)\,\mathrm{d}\mu$$

$$\mathrm{II}^{(0)}(\nu) = \int_0^1 W^{(0)}(\mu)\phi(\nu, \mu)\,\mathrm{d}\mu$$

to be

$$\mathrm{I}_{+}^{(0)} = \mathscr{F}_0\left\{\frac{c\nu}{4}\left[\frac{\nu_0-\alpha}{\nu_0+\nu}\ln\left(\frac{\nu_0+1}{\nu_0}\right)+\frac{\nu_0+\alpha}{\nu_0-\nu}\ln\left(\frac{\nu_0-1}{\nu_0}\right)\right]\right.$$
$$\left.+\frac{\nu(\alpha+\nu)}{\nu_0^2-\nu^2}\left[1-\frac{c\nu}{2}\ln\left(\frac{1+\nu}{\nu}\right)\right]\right\}$$

$$\mathrm{I}_{-}^{(0)} = \mathscr{F}_0 \left\{ \frac{c\nu}{2(\nu_0+\nu)^2} [\alpha\nu - \nu_0(2\nu+\nu_0)] \left[\ln\left(\frac{\nu_0+1}{\nu_0}\right) - \frac{1}{\nu_0+1} \right] + \frac{\nu(\alpha+\nu)}{(\nu_0+\nu)^2} \left[1 - \frac{c\nu}{2} \ln\left(\frac{1+\nu}{\nu}\right) - \frac{c\nu_0}{2} \frac{1}{\nu_0+1} \right] \right\}$$

$$\mathrm{II}^{(0)} = \frac{c\nu}{2\Omega^{(0)}(1-c)} \left\{ \frac{c}{2} \left[\frac{\nu_0(\nu_0-\alpha)}{\nu_0+\nu} \ln\left(\frac{\nu_0+1}{\nu_0}\right) - 1 \right] + \frac{\nu_0+\alpha}{\nu_0+\nu} \left[1 - \frac{c\nu}{2} \ln\left(\frac{1+\nu}{\nu}\right) \right] \right\}$$

where $\mathscr{F}_0 = c^2\nu_0/4\Omega^{(0)}(1-c)$.

Comparison of $\mathrm{I}_{\pm}^{(0)}$ and $\mathrm{II}^{(0)}$ with the true values of these integrals WRT $W(\mu)$ (namely 0, $c^2\nu_0^2\nu X(-\nu_0)/2(\nu_0+\nu)$ and $c\nu/2$ respectively) is shown in table 2. Together with the example of [4], this demonstrates that the simple $W^{(0)}(\mu)$ can be used as the weight function in half-range problems with a fair degree of confidence; see, however, § 4 below for further discussions on this.

2.4. The quadrature nodes

The collocation points for use in (2.5) are taken to be the zeros of the polynomials $C_N(\nu)$ [6]—the set of orthogonal polynomials in (0, 1) WRT $W^{(0)}(\mu)$. Table 3 shows the zeros for some values of c and $N = 1, 2, 3, 4, 5$.

2.5. The parameter ε

So far, ε has been a convergence parameter in $\phi_\varepsilon(\nu, \mu)$. In order to be usable in the discretised spectral, or $\Delta\sigma_N$ approximation, it must be a finite and small positive quantity, since as $\varepsilon \to 0$ the Poisson and conjugate Poisson kernels must distributionally converge to their respective GF. It is also necessary, from § 2.2, that

$$\varepsilon \to 0 \qquad \text{as} \qquad N \to \infty.$$

An acceptable semi-empirical ε-N relationship is obtained by noting that Poisson representation of the delta function $\delta_\varepsilon(x) = \varepsilon/\pi(x^2+\varepsilon^2)$ tends to infinity at $x = 0$ as $1/\varepsilon\pi$. Setting this equal to N, the number of transient terms in (2.5), gives the required relationship to be

$$\varepsilon = (N\pi)^{-1}. \tag{2.15}$$

For a particular N ($N = 0 \Rightarrow$ no transients), (2.15) gives the corresponding ε. Equation (2.15) can be improved from first principles. This definition, not used in this work, is given in the appendix.

2.6. The $\Delta\sigma_N$ approximation

With the different components of the $\Delta\sigma_N$ approximation introduced above, it follows that the $\Delta\sigma_N$ solution of the transport equation can be expressed as given by (2.5). The superposition coefficients $\{a(\nu_i)\}_0^N$ are to be obtained from a half-range boundary condition of the form

$$f(\mu) = a(\nu_0)\, \mathrm{e}^{-x_0/\nu_0} \phi(\nu_0, \mu) + \sum_{i=1}^{N} a(\nu_i)\, \mathrm{e}^{-x_0/\nu_i} \phi_\varepsilon(\nu_i, \mu) \tag{2.16}$$

Table 2. Difference of the integrals $I_{\pm}^{(0)}$, $II^{(0)}$ from their true values $I_{\pm}$, II.

c	ν	$\lvert I_{+}^{(0)}(\nu)\rvert$	$\lvert I_{-}^{(0)}(\nu) - I_{-}(\nu)\rvert$	$\lvert II^{(0)}(\nu) - II(\nu)\rvert$
0.2	0.1	0.535 385 (−5)	0.572 870 (−5)	0.469 284 (−4)
	0.2	0.707 294 (−6)	0.294 326 (−5)	0.314 496 (−5)
	0.3	0.662 996 (−5)	0.147 796 (−6)	0.501 818 (−4)
	0.4	0.146 901 (−4)	0.205 905 (−5)	0.931 733 (−4)
	0.5	0.227 235 (−4)	0.263 190 (−5)	0.119 912 (−3)
	0.6	0.304 229 (−4)	0.204 555 (−5)	0.129 249 (−3)
	0.7	0.376 692 (−4)	0.542 429 (−6)	0.121 829 (−3)
	0.8	0.444 289 (−4)	0.165 289 (−5)	0.989 440 (−4)
	0.9	0.507 083 (−4)	0.435 392 (−5)	0.620 606 (−4)
0.4	0.1	0.442 137 (−4)	0.396 535 (−4)	0.191 159 (−3)
	0.2	0.447 175 (−5)	0.919 042 (−5)	0.171 081 (−5)
	0.3	0.561 205 (−4)	0.219 256 (−4)	0.221 993 (−3)
	0.4	0.121 524 (−3)	0.421 658 (−4)	0.400 548 (−3)
	0.5	0.185 919 (−3)	0.507 541 (−4)	0.512 081 (−3)
	0.6	0.247 063 (−3)	0.494 671 (−4)	0.553 502 (−3)
	0.7	0.304 177 (−3)	0.404 587 (−4)	0.528 412 (−3)
	0.8	0.357 112 (−3)	0.256 554 (−4)	0.442 622 (−3)
	0.9	0.406 051 (−3)	0.662 701 (−5)	0.302 376 (−3)
0.6	0.1	0.157 646 (−3)	0.126 764 (−3)	0.463 714 (−3)
	0.2	0.359 964 (−5)	0.609 907 (−5)	0.183 628 (−4)
	0.3	0.220 750 (−3)	0.138 257 (−3)	0.577 815 (−3)
	0.4	0.457 912 (−3)	0.228 937 (−3)	0.102 877 (−2)
	0.5	0.688 198 (−3)	0.276 101 (−3)	0.132 375 (−2)
	0.6	0.904 600 (−3)	0.286 580 (−3)	0.145 834 (−2)
	0.7	0.110 509 (−2)	0.268 323 (−3)	0.144 268 (−2)
	0.8	0.128 971 (−2)	0.228 385 (−3)	0.129 125 (−2)
	0.9	0.145 935 (−2)	0.172 483 (−3)	0.101 898 (−2)
0.8	0.1	0.455 944 (−3)	0.384 454 (−3)	0.104 659 (−2)
	0.2	0.311 475 (−4)	0.440 392 (−4)	0.913 965 (−4)
	0.3	0.711 181 (−3)	0.490 447 (−3)	0.143 592 (−2)
	0.4	0.141 645 (−2)	0.826 292 (−3)	0.258 405 (−2)
	0.5	0.209 275 (−2)	0.103 920 (−2)	0.343 496 (−2)
	0.6	0.272 244 (−2)	0.114 477 (−2)	0.397 879 (−2)
	0.7	0.330 162 (−2)	0.116 362 (−2)	0.423 548 (−2)
	0.8	0.383 184 (−2)	0.111 481 (−2)	0.422 360 (−2)
	0.9	0.431 668 (−2)	0.101 429 (−2)	0.400 246 (−2)

$f(\mu)$, $\mu > 0$, given. Either the Mark (collocation) or Marshak (Galerkin) type of condition may now be imposed on the residual obtained from (2.16) for a finite number of terms N in the sum. The simpler collocation procedure uses the $N+1$ zeros of $C_{N+1}(\mu)$ to form the $N+1$ equations for the $a(\nu_i)$, $i = 0, 1, 2, \ldots, N$. For not too large N, or when $f(\mu)$ is either a constant or has a singular form (as in the albedo problem, for example), the algebraically more complicated Galerkin procedure WRT $W^{(0)}(\mu)$ as the weight function is to be used. This requires the evaluation of the following integrals:

$$I_{\pm}^{(0)}(\nu) \qquad I_{\varepsilon+}^{(0)}(\nu) = \int_0^1 W^{(0)}(\mu)\phi(\nu_0, \mu)\phi_{\varepsilon}(\nu, \mu)\, d\mu$$

$$III_{\varepsilon}^{(0)}(\nu_i, \nu_j) = \int_0^1 W^{(0)}(\mu)\phi_{\varepsilon}(\nu_i, \mu)\phi(\nu_j, \mu)\, d\mu.$$

We have

$$
\begin{aligned}
\mathrm{I}_{\varepsilon+}^{(0)}(\nu) = {} & \frac{\mathscr{F}_0}{2}\left(\frac{c\nu}{2}(\nu_0+\nu)+\frac{\lambda_\varepsilon}{\pi_\varepsilon}\varepsilon\right)\frac{\nu_0-\alpha}{(\nu_0+\nu)^2+\varepsilon^2}\ln\left(\frac{\nu_0+1}{\nu_0}\right) \\
& + \frac{\mathscr{F}_0}{2}\left(\frac{c\nu}{2}(\nu_0-\nu)-\frac{\lambda_\varepsilon}{\pi_\varepsilon}\varepsilon\right)\frac{\nu_0+\alpha}{(\nu_0-\nu)^2+\varepsilon^2}\ln\left(\frac{\nu_0-1}{\nu_0}\right) \\
& + \frac{\mathscr{F}_0}{\Delta_0}\left[\frac{c\nu}{4}[(\nu^2+\varepsilon^2)^2-\nu_0^2(\nu^2-\varepsilon^2)-\alpha\nu(\nu_0^2-\nu^2-\varepsilon^2)]\right. \\
& \left. + \frac{\lambda_\varepsilon}{\pi_\varepsilon}\varepsilon\left(\nu\nu_0^2+\frac{\alpha}{2}(\nu_0^2+\nu^2+\varepsilon^2)\right)\right] L_\varepsilon(\nu) \\
& + \frac{\mathscr{F}_0}{\Delta_0}\left[c\nu\varepsilon\left(\nu\nu_0^2+\frac{\alpha}{2}(\nu_0^2+\nu^2+\varepsilon^2)\right)\right. \\
& \left. + \frac{\lambda_\varepsilon}{\pi_\varepsilon}[\nu_0^2(\nu^2-\varepsilon^2)-(\nu^2+\varepsilon^2)^2+\alpha\nu(\nu_0^2-\nu^2-\varepsilon^2)]\right] T_\varepsilon(\nu) \qquad (2.17)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_0 &= [(\nu_0+\nu)^2+\varepsilon^2][(\nu_0-\nu)^2+\varepsilon^2] \\
L_\varepsilon(\nu) &= \ln\left(\frac{(1-\nu)^2+\varepsilon^2}{\nu^2+\varepsilon^2}\right) \\
T_\varepsilon(\nu) &= \tan^{-1}\left(\frac{\nu}{\varepsilon}\right)+\tan^{-1}\left(\frac{1-\nu}{\varepsilon}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{III}_\varepsilon^{(0)}(\nu_i,\nu_j) = {} & \mathscr{F}_j\left(\frac{c\nu_i}{2}(\nu_0+\nu_i)+\frac{\lambda_\varepsilon}{\pi_\varepsilon}\varepsilon\right)\frac{\nu_0(\nu_0-\alpha)}{(\nu_0+\nu_j)[(\nu_0+\nu_i)^2+\varepsilon^2]}\ln\left(\frac{\nu_0+1}{\nu_0}\right) \\
& + \mathscr{F}_j\left(\frac{c\nu_i}{2}(\nu_j-\nu_i)-\frac{\lambda_\varepsilon}{\pi_\varepsilon}\varepsilon\right)\frac{\nu_j(\alpha+\nu_j)}{(\nu_0+\nu_j)[(\nu_i-\nu_j)^2+\varepsilon^2]}\ln\left(\frac{1-\nu_j}{\nu_j}\right) \\
& + \frac{\mathscr{F}_j}{\Delta_{ij}}\left(\frac{c\nu_i}{4}[(\nu_i^2+\varepsilon^2)^2-\nu_0\nu_j(\nu_i^2-\varepsilon^2)\right. \\
& - \alpha\nu_i(\nu_0\nu_j-\nu_i^2-\varepsilon^2)-(\nu_j-\nu_0)(\nu_i^2+\varepsilon^2)(\alpha+\nu_i)] \\
& \left. + \frac{\lambda_\varepsilon}{\pi_\varepsilon}\varepsilon[\nu_0\nu_i\nu_j+\tfrac{1}{2}\alpha(\nu_i^2+\nu_0\nu_j+\varepsilon^2)+\tfrac{1}{2}(\nu_j-\nu_0)(\nu_i^2+\varepsilon^2)]\right) L_\varepsilon(\nu_i) \\
& + \frac{\mathscr{F}_j}{\Delta_{ij}}\left(c\nu_i\varepsilon[\nu_0\nu_i\nu_j+\tfrac{1}{2}\alpha(\nu_i^2+\nu_0\nu_j+\varepsilon^2)+\tfrac{1}{2}(\nu_j-\nu_0)(\nu_i^2+\varepsilon^2)]\right. \\
& + \frac{\lambda_\varepsilon}{\pi_\varepsilon}[-(\nu_i^2+\varepsilon^2)^2+\nu_0\nu_j(\nu_i^2-\varepsilon^2)+\alpha\nu_i(\nu_0\nu_j-\nu_i^2-\varepsilon^2) \\
& \left. + (\alpha+\nu_i)(\nu_j-\nu_0)(\nu_i^2+\varepsilon^2)]\right) T_\varepsilon(\nu_i) \\
& + \mathscr{F}\lambda(\nu_j)\frac{\nu_j(\alpha+\nu_j)}{(\nu_0+\nu_j)[(\nu_i-\nu_j)^2+\varepsilon^2]}\left(\frac{c\nu_i}{2}(\nu_i-\nu_j)+\frac{\lambda_\varepsilon}{\pi_\varepsilon}\varepsilon\right) \qquad (2.18)
\end{aligned}
$$

where

$$\mathcal{F} = c/2\Omega^{(0)}(1-c) \qquad \mathcal{F}_j = c^2\nu_j/4\Omega^{(0)}(1-c)$$

$$\Delta_{ij} = [(\nu_0+\nu_i)^2+\varepsilon^2][(\nu_i-\nu_j)^2+\varepsilon^2].$$

Figure 1 shows the Poisson and conjugate Poisson kernels for some ε values, while figures 2 and 3 plot the $\Delta\sigma_1$ and $\Delta\sigma_2$ approximations of $\phi(\nu,\mu)$. Here and in the following the discretised spectrum is ordered according to $\nu_0 > \nu_1 > \nu_2 > \cdots > \nu_N$.

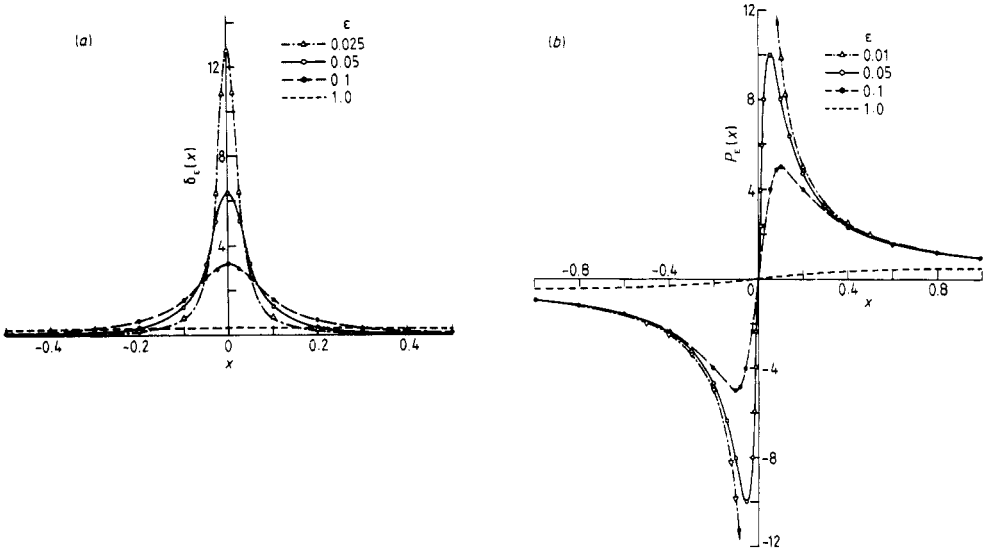

Figure 1. (*a*) The Poisson kernel $\delta_\varepsilon(x)$. (*b*) The conjugate Poisson kernel $P_\varepsilon(x)$. The maximum and minimum values of $P_\varepsilon(x)$ are ± 50 for $\varepsilon = \pm 0.01$.

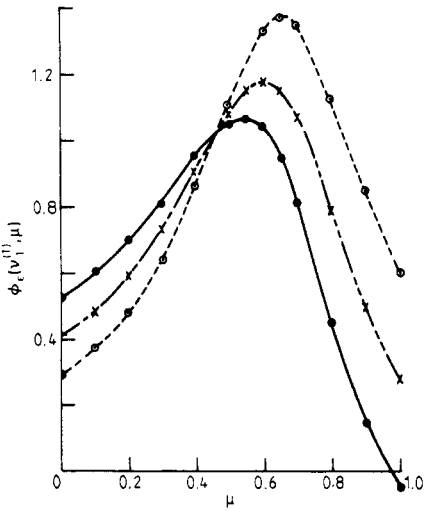

Figure 2. $\Delta\sigma_1$ approximation of the singular eigenfunction, for $\varepsilon = 1/\pi$ and values of: ⊙ $c = 0.1$, $\nu_1 = 0.668$; × $c = 0.5$, $\nu_1 = 0.675$; ● $c = 0.9$, $\nu_1 = 0.691$.

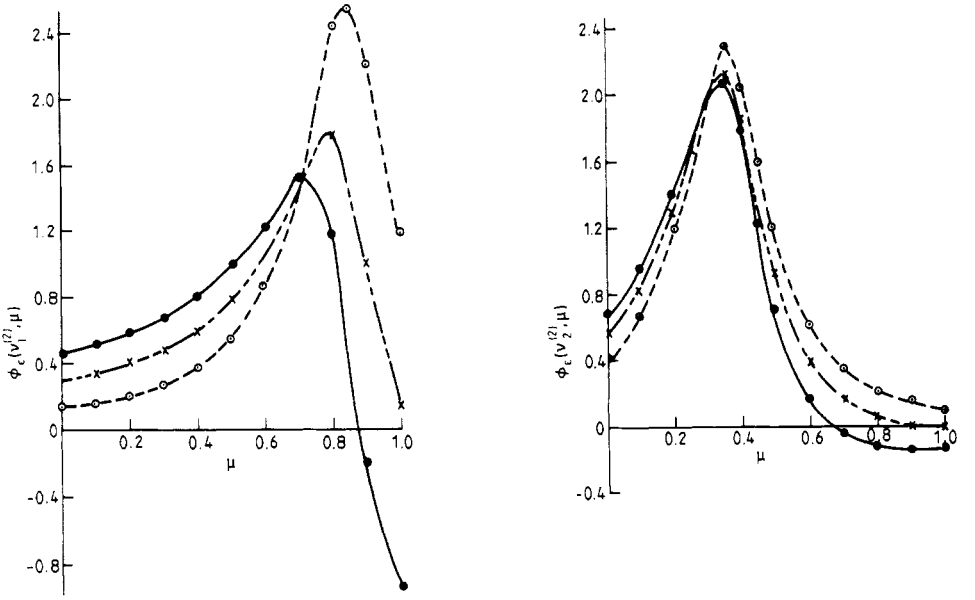

Figure 3. $\Delta\sigma_2$ approximations of the singular eigenfunction, for $\varepsilon = 1/2\pi$ and values of: ⊙ $c = 0.1$, $\nu_1 = 0.845$, $\nu_2 = 0.356$; × $c = 0.5$, $\nu_1 = 0.848$, $\nu_2 = 0.362$; ● $c = 0.9$, $\nu_1 = 0.854$, $\nu_2 = 0.377$.

3. Sample numerical result

As it is not the purpose of the present paper to report on the details of the numerical adaption of the theory above, this section contains a sample illustration of the $\Delta\sigma_1$ and $\Delta\sigma_2$ calculations for the emergent angular distribution and leakage of the standard Milne problem, using the collocation method for the residual. These results are therefore not intended to be definitive, and details of the Galerkin calculations will be reported separately. For this problem, the $\Delta\sigma_N$ approximation takes the form

$$\psi_N(x, \mu) = a_\varepsilon(\nu_0)\, e^{-x/\nu_0}\phi(\nu_0, \mu) + e^{x/\nu_0}\phi(-\nu_0, \mu) + \sum_{i=1}^{N} a_\varepsilon(\nu_i)\, e^{-x/\nu_i}\phi_\varepsilon(\nu_i, \mu)$$

where the coefficients $\{a_\varepsilon(\nu_i)\}_0^N$ are obtained from the equation

$$-\phi(-\nu_0, \mu_j) = a_\varepsilon(\nu_0)\phi(\nu_0, \mu_j) + \sum_{i=1}^{N} a_\varepsilon(\nu_i)\phi_\varepsilon(\nu_i, \mu_j) \qquad j = 1, 2, \ldots, N+1$$

where $C_N(\nu_i) = 0$ and $C_{N+1}(\mu_j) = 0$. Then the emergent distribution is

$$\frac{2}{c}\psi_N(0, -\mu) = a_\varepsilon(\nu_0)\frac{\nu_0}{\nu_0+\mu} + \frac{\nu_0}{\nu_0-\mu} + \sum_{1}^{N} a_\varepsilon(\nu_i)\frac{\nu_i}{\nu_i+\mu} \qquad \mu > 0$$

and the expression for leakage takes the form

$$\frac{2}{c}J_N = -a_\varepsilon(\nu_0)\nu_0\left[1 - \nu_0 \ln\left(\frac{\nu_0+1}{\nu_0}\right)\right] + \nu_0\left[1 - \nu_0 \ln\left(\frac{\nu_0}{\nu_0-1}\right)\right]$$
$$-\sum_{1}^{N} a_\varepsilon(\nu_i)\nu_i\left[1 - \nu_i \ln\left(\frac{\nu_i+1}{\nu_i}\right)\right].$$

Table 3. Zeros of the polynomials $C_N(\mu)$, $N = 1(1)5$.

c	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
0.1	0.667 877 00	0.356 099 81	0.212 988 67	0.140 158 06	0.098 790 05
		0.845 386 45	0.591 242 24	0.417 053 27	0.305 045 09
			0.911 608 37	0.723 591 06	0.562 517 54
				0.942 999 13	0.802 261 18
					0.960 250 82
0.2	0.669 245 24	0.377 302 60	0.213 737 53	0.140 619 96	0.099 086 44
		0.845 880 39	0.592 049 05	0.417 790 14	0.305 630 81
			0.911 830 09	0.724 083 58	0.563 078 67
				0.943 115 85	0.802 572 37
					0.960 319 39
0.3	0.670 810 50	0.358 700 12	0.214 614 75	0.141 163 34	0.099 435 94
		0.846 444 94	0.592 978 71	0.418 644 74	0.306 313 38
			0.912 083 59	0.724 649 48	0.563 726 45
				0.943 249 36	0.802 929 49
					0.960 397 84
0.4	0.672 619 46	0.360 337 66	0.215 649 98	0.141 806 94	0.099 850 70
		0.847 097 47	0.594 060 66	0.419 644 69	0.307 115 21
			0.912 376 84	0.725 306 76	0.564 481 67
				0.943 403 90	0.803 343 97
					0.960 488 70
0.5	0.674 736 59	0.362 273 12	0.216 879 33	0.142 572 89	0.100 344 76
		0.847 862 45	0.595 334 56	0.420 825 74	0.308 064 36
			0.912 721 14	0.726 080 26	0.565 372 17
				0.943 585 52	0.803 831 80
					0.960 595 55
0.6	0.677 267 61	0.364 599 37	0.218 359 71	0.143 495 54	0.100 939 74
		0.848 780 06	0.596 864 91	0.422 245 34	0.309 205 40
			0.913 135 04	0.727 010 57	0.566 443 14
				0.943 804 17	0.804 419 13
					0.960 724 31
0.7	0.680 403 12	0.367 483 08	0.220 192 52	0.144 835 89	0.101 673 87
		0.849 922 76	0.598 767 80	0.424 006 55	0.310 618 01
			0.913 652 07	0.728 170 84	0.567 775 79
				0.944 077 81	0.805 153 21
					0.960 885 66
0.8	0.684 536 32	0.371 266 09	0.222 584 47	0.146 117 61	0.102 624 46
		0.851 440 60	0.601 282 56	0.426 320 75	0.312 464 92
			0.914 341 84	0.729 712 49	0.569 537 47
				0.944 443 85	0.806 132 04
					0.961 101 86
0.9	0.690 738 59	0.376 870 17	0.226 087 94	0.148 269 28	0.103 996 01
		0.853 745 28	0.605 060 32	0.429 758 21	0.315 182 12
			0.915 396 17	0.732 050 00	0.572 182 95
				0.945 005 61	0.807 625 07
					0.961 434 56

Table 4. Values of the asymptotic coefficient in the source-free Milne problem.

c	$N=1$, $\varepsilon=0.3183$	$N=2$, $\varepsilon=0.1592$	Exact
0.2	0.0705	0.0199	−0.0004
0.4	0.0254	−0.0044	−0.0273
0.6	−0.0240	−0.0638	−0.1149
0.8	−0.1517	−0.2280	−0.2827
0.9	−0.3065	−0.3914	−0.4362

Table 5. Emergent angular distribution in the source-free Milne problem.

	$c=0.2$			$c=0.4$		
$-\mu$	$N=1$	$N=2$	Exact	$N=1$	$N=2$	Exact
1.0	1100.37	1100.37	1098.62	13.8992	13.8990	13.8823
0.8	0.4987	0.4972	0.4965	0.9310	0.9310	0.9324
0.6	0.2486	0.2468	0.2466	0.4725	0.4723	0.4750
0.4	0.1648	0.1628	0.1628	0.3099	0.3095	0.3133
0.2	0.1224	0.1199	0.1202	0.2239	0.2230	0.2281
0.0	0.0961	0.0925	0.0926	0.1669	0.1638	0.1686
	$c=0.6$			$c=0.8$		
$-\mu$	$N=1$	$N=2$	Exact	$N=1$	$N=2$	Exact
1.0	3.2032	3.2035	3.2066	1.3033	1.2998	1.3019
0.8	1.0557	1.0560	1.0603	0.8398	0.8362	0.8393
0.6	0.6139	0.6143	0.6197	0.5989	0.5954	0.5999
0.4	0.4183	0.4185	0.4256	0.4456	0.4420	0.4486
0.2	0.3023	0.3018	0.3113	0.3323	0.3281	0.3378
0.0	0.2176	0.2135	0.2225	0.2349	0.2268	0.2359

Table 6. Leakage in the source-free Milne problem.

	$c=0.2$	$c=0.4$	$c=0.6$	$c=0.8$
$N=1$	0.8301	0.6621	0.5142	0.3709
$N=2$	0.8292	0.6619	0.5143	0.3690
Exact	0.8280	0.6627	0.5170	0.3713

Tables 4–6 show the results for $N=1, 2$. Table 4 compares $a_\varepsilon(\nu_0)$ with the exact $a(\nu_0) = -\exp(-2z_0/\nu_0)$ and shows that $a_{1/2\pi}(\nu_0)$ is more accurate than $a_{1/\pi}(\nu_0)$ in all the cases.

4. Discussion

In this paper, we have shown how the singular eigenfunction method can be formulated in terms of regular functions which tend, as a parameter ε in them approaches 0, to

the singular functions. This is achieved by approximating the singular eigenfunctions by a rational function rather than by the usual polynomial series. Discretisation of the continuous spectrum $(0, 1)$ is done through the zeros of the polynomials orthogonal in $(0, 1)$ WRT the weight $W^{(0)}(\mu)$ [6]. It has been our experience that attempts to improve upon this simple one-parameter weight through more adjustable parameters are largely unsuccessful in the whole μ interval. However, the zeros of $C_N(\mu)$ are the approximate Gaussian nodes for the half-range problem and therefore enjoy a special status as compared to, for example, equally spaced points which are not Gaussian for any weight.

In our opinion, a more basic definition of ε than (2.15) is necessary. For example, an arbitrary smaller ε for a given N can lead to worsening of the numerical results. This is supports our hypothesis that ε should be N dependent, and a definition using the concept of equivalent Cauchy sequences for the delta function is given in the appendix. Use of this definition in a Galerkin treatment of the residual is the subject of a separate publication [9].

Appendix. A fundamental definition of ε

The definition of ε contained in this appendix is based on the idea of equivalent Cauchy sequences in a metric space. We recall that (i) two Cauchy sequences $\{x_k\}$ and $\{y_k\}$ in a metric space (X, d) are said to be equivalent if $d(x_k, y_k) \to 0$, $k \to \infty$, and (ii) an equivalence class $\mathscr{X}$ consists of all Cauchy sequences equivalent to a given Cauchy sequence $\{x_k\}$. In this context, a distribution or generalised function can be defined as the common limit $\mathscr{X}$ of an equivalence class of locally integrable functions (i.e. it is an element of the completion of the space of locally integrable functions) with convergence defined WRT the metric

$$d(x_i, x_j) = \int_I (x_i(t) - x_j(t))\varphi(t)\,\mathrm{d}t \qquad \varphi(t) \in C_0^\infty(I).$$

In this new definition of ε, we consider the first two among the following examples of equivalent delta convergent Cauchy sequences:

$$\delta_N^{(1)}(t) = \begin{cases} N & -1/2N \leqslant t \leqslant 1/2N \\ 0 & \text{otherwise} \end{cases} \qquad N = 0, 1, 2, \ldots \tag{A1}$$

$$\delta_\varepsilon^{(2)}(t) = \varepsilon/\pi(t^2 + \varepsilon^2) \qquad \varepsilon \to 0 \tag{A2}$$

$$\delta_n^{(3)}(t) = \sin(nt)/\pi t \qquad n \to \infty \tag{A3}$$

$$\delta_\varepsilon^{(4)}(t) = \exp(-t^2/\varepsilon)/(\pi\varepsilon)^{1/2} \qquad \varepsilon \to 0 \tag{A4}$$

$$\delta_n^{(5)}(t) = n \exp(-n|t|)/2 \qquad n \to \infty \tag{A5}$$

and note that (A1) and (A2) are normalised to unity in the intervals $(-1/2N, 1/2N)$ and $(-\infty, \infty)$ respectively. In terms of these, $\phi(\nu, \mu)$, $-1 \leqslant \nu \leqslant 1$, has the following equivalent representations:

$$\phi_N(\nu, \mu) = \frac{c\nu}{2} \frac{\nu - \mu}{(\nu - \mu)^2 + \varepsilon^2} + \lambda_\varepsilon(\nu) N \qquad \nu \pm 1/2N \lesseqgtr \pm 1;\ N = 1, 2, 3, \ldots \tag{A6}$$

and

$$\phi_\varepsilon(\nu, \mu) = \frac{c\nu}{2} \frac{\nu - \mu}{(\nu - \mu)^2 + \varepsilon^2} + \frac{\lambda_\varepsilon}{\pi_\varepsilon} \frac{\varepsilon}{(\nu - \mu)^2 + \varepsilon^2}. \tag{A7}$$

If ν and N in (A6) do not satisfy the restrictions shown there, N is to be replaced by N^* such that normalisation again holds and in this case, one has

$$\phi_{N^*}(\nu, \mu) = \frac{c\nu}{2} \frac{\nu - \mu}{(\nu - \mu)^2 + \varepsilon^2} + \lambda_\varepsilon(\nu) N^*$$

where N^* is given by

$$N^*(\nu) = \begin{cases} (1 + \nu + 1/2N)^{-1} & \nu \mp 1/2N < \mp 1 \quad \text{(A8)} \\ (1 - \nu + 1/2N)^{-1} & \nu \pm 1/2N > \pm 1 \quad \text{(A9)} \end{cases}$$

such that

$$\int_{-1}^{\nu+1/2N} \delta^{(1)}_{N^*}(\nu - \mu)\, \mathrm{d}\mu = 1 \qquad \int_{\nu-1/2N}^{1} \delta^{(1)}_{N^*}(\nu - \mu)\, \mathrm{d}\mu = 1.$$

With the ν discretised at the zeros of C_N, only case (A9) can arise for $\nu = \nu_1$; for all other $\nu = \nu_2, \nu_3, \ldots, \nu_N$, $N^* = N$. The definition of ε is now obtained by setting

$$\phi_\varepsilon(\nu, \nu) = \phi_{N^*}(\nu, \nu).$$

This gives the fundamental ε-N relationship to be

$$\varepsilon(N, \nu) = (N^* \pi_\varepsilon)^{-1} \quad \text{(A10)}$$

where

$$\pi_\varepsilon = \tan^{-1}\left(\frac{1+\nu}{\varepsilon}\right) + \tan^{-1}\left(\frac{1-\nu}{\varepsilon}\right)$$

so as to normalise (A2) in $(-1, 1)$. Expression (2.15) used in the main text, is an approximation of (A10) when both (A1) and (A2) are normalised in $(-\infty, \infty)$ instead of $(-1, 1)$. An effective iteration scheme for (A10) is: $\varepsilon_{i+1} = (N^* \pi_{\varepsilon_i})^{-1}$, $\varepsilon_0 = (N^* \pi)^{-1}$. We remark that the reason why point equivalence rather than distributional equivalence of ϕ_ε and ϕ_{N^*} has been employed in the derivation of (A10) is that in the later use the distinctive character of the Poisson kernel would be largely lost in favour of the representation (A1).

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